廣義相對論的作用量

萬思揚

May 21, 2022

目錄

1 作用量

廣義相對論中的作用量分爲三個部分：第一，時空的作用量，即 Einstein-Hilbert 作用量；第二，物質場的作用量；第三，邊界項，其中 Gibbons-Hawking-York 邊界項是比較常用的邊界項。

$$S = S_{EH} + S_M + S_B \tag{1.1}$$

本章節先討論作用量的前兩個部分，其中，我們討論更一般的帶宇宙學常數 Λ 的情況，

$$\begin{cases} S_{EH} = \dfrac{1}{2\kappa}\displaystyle\int (R - 2\Lambda)\, \epsilon \\ S_M = \displaystyle\int \mathcal{L}_M\, \epsilon \end{cases} \tag{1.2}$$

1.1 Einstein-Hilbert 作用量的變分

Eistein-Hilbert 作用量的變分包含體元 ϵ 的變分和 Ricci 標量 R 的變分。

首先考慮 ϵ 的變分，

$$\epsilon = \sqrt{g}\, (dx^1)_{a_1} \wedge ... \wedge (dx^n)_{a_n} \tag{1.3}$$

對於 g，

$$
\begin{aligned}
\delta g &= \mathrm{sgn}(\det g)\,\frac{\partial}{\partial g_{\mu\nu}}\Big(\det\{g_{\rho\sigma}\}\Big)\,\delta g_{\mu\nu} \\
&= \mathrm{sgn}(\det g)\det\{g_{\rho\sigma}\}g^{\mu\nu}\delta g_{\mu\nu}
\end{aligned} \tag{1.4}
$$

注意到，

$$
\begin{aligned}
& g_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} = 0 \\
\Longrightarrow\, & g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}
\end{aligned} \tag{1.5}
$$

更一般地，還有，

$$
\begin{aligned}
& g_{\mu\nu}\delta g^{\nu\rho} + g^{\nu\rho}\delta g_{\mu\nu} = 0 \\
\Longrightarrow\, & \delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}
\end{aligned} \tag{1.6}
$$

所以，可以認爲，

$$
\frac{\partial g_{\mu\nu}}{\partial g^{\rho\sigma}} = -g_{\mu\rho}g_{\nu\sigma} \quad \text{實際上，} g_{\mu\nu} = \frac{G^{\mu\nu}}{\det\{g^{\rho\sigma}\}} \tag{1.7}
$$

其中 $G^{\mu\nu}$ 是方陣 $\{g^{\mu\nu}\}$ 中的元素 $g^{\mu\nu}$ 的代數餘子式。將(1.5)代入(1.4)，

$$
\delta g = -g\, g_{\mu\nu}\delta g^{\mu\nu} \tag{1.8}
$$

再代入(1.3)，得到適配體元的變分，

$$
\delta\epsilon = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\epsilon = -\frac{1}{2}g_{ab}\delta g^{ab}\epsilon \tag{1.9}
$$

再考慮 δR，

$$
\delta R = R_{ab}\delta g^{ab} + g^{ab}\delta R_{ab} \tag{1.10}
$$

其中，在任意參考系 $\{x^\mu\}$ 下，

$$
\begin{aligned}
\delta R_{ab} = \delta R^c{}_{acb} =& \delta\left(2\partial_{[c}\Gamma^c{}_{b]a} + 2\Gamma^c{}_{d[c}\Gamma^d{}_{b]a}\right) \\
=& \partial_c\delta\Gamma^c{}_{ba} - \partial_b\delta\Gamma^c{}_{ca} \\
& + (\delta\Gamma^c{}_{dc})\Gamma^d{}_{ba} + \Gamma^c{}_{dc}\delta\Gamma^d{}_{ba} - (\delta\Gamma^c{}_{db})\Gamma^d{}_{ca} - \Gamma^c{}_{db}\delta\Gamma^d{}_{ca} \quad \text{注意到，}\Gamma^c{}_{db}\delta\Gamma^d{}_{ca} = \Gamma^d{}_{cb}\delta\Gamma^c{}_{da} \\
=& \left(\partial_c\delta\Gamma^c{}_{ba} + \Gamma^c{}_{dc}\delta\Gamma^d{}_{ba} - \Gamma^d{}_{ca}\delta\Gamma^c{}_{db} - \Gamma^d{}_{cb}\delta\Gamma^c{}_{ad}\right) - \left(\partial_b\delta\Gamma^c{}_{ca} - \Gamma^d{}_{ba}\delta\Gamma^c{}_{cd}\right) \\
=& \nabla_c\delta\Gamma^c{}_{ba} - \nabla_b\delta\Gamma^c{}_{ca}
\end{aligned} \tag{1.11}
$$

代入(1.10)，

$$
\begin{aligned}
\delta R &= R_{ab}\delta g^{ab} + g^{ab}\left(\nabla_c\delta\Gamma^c{}_{ba} - \nabla_b\delta\Gamma^c{}_{ca}\right) \\
&= R_{ab}\delta g^{ab} + \nabla_a\left(g^{bc}\delta\Gamma^a{}_{bc} - g^{ab}\delta\Gamma^c{}_{cb}\right)
\end{aligned} \tag{1.12}
$$

將(1.9)、(1.12)代入 Einstein-Hilbert 作用量，得到，

$$
\begin{aligned}
\delta S_{EH} &= \frac{1}{2\kappa}\int\left(\delta R\,\epsilon + (R-2\Lambda)\,\delta\epsilon\right) \\
&= \frac{1}{2\kappa}\int\left(R_{ab}\delta g^{ab} + \nabla_a\left(g^{bc}\delta\Gamma^a{}_{bc} - g^{ab}\delta\Gamma^c{}_{cb}\right) - \frac{1}{2}(R-2\Lambda)\,g_{ab}\delta g^{ab}\right)\epsilon \\
&= \frac{1}{2\kappa}\int\left(R_{ab} - \frac{1}{2}g_{ab}R + \Lambda\right)\delta g^{ab}\,\epsilon + \frac{1}{2\kappa}\int\nabla_a\left(g^{bc}\delta\Gamma^a{}_{bc} - g^{ab}\delta\Gamma^c{}_{cb}\right)\epsilon
\end{aligned} \tag{1.13}
$$

可以看到，在(1.13)中，就已經出現了一個邊界項。它是一個全散度的積分，但是與一般的邊界項不同，它并不等於零，這將在第 2 章中予以討論。

1.2 物質部分作用量的變分

物質場作用量的變分，

$$\delta S_M = \int \left(\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} \delta g^{\mu\nu}\,\epsilon + \mathcal{L}_M\,\delta\epsilon \right) \tag{1.14}$$

代入(1.9)，得到，

$$\delta S_M = \int \left(\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{L}_M \right) \delta g^{\mu\nu}\,\epsilon \tag{1.15}$$

注意到，

$$\begin{aligned} g^{-1}\, g_{\mu\nu} &= \frac{\partial g^{-1}}{\partial g^{\mu\nu}} \\ &= -\frac{1}{g^2} \frac{\partial g}{\partial g^{\mu\nu}} \\ \implies \qquad g_{\mu\nu} &= -\frac{1}{g} \frac{\partial g}{\partial g^{\mu\nu}} \end{aligned} \tag{1.16}$$

這裏我們可以驗證(1.7)，

$$\frac{\partial g}{\partial g^{\mu\nu}} = \frac{\partial g_{\rho\sigma}}{\partial g^{\mu\nu}} \frac{\partial g}{\partial g_{\rho\sigma}} = (-g_{\rho\mu} g_{\sigma\nu})\, g\, g^{\rho\sigma} = -g\, g_{\mu\nu} \tag{1.17}$$

將(1.16)代入(1.15)，

$$\begin{aligned} \delta S_M &= \int \left(\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} + \frac{1}{2g} \frac{\partial g}{\partial g^{\mu\nu}} \mathcal{L}_M \right) \delta g^{\mu\nu}\,\epsilon \\ &= \int \left(\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial g^{\mu\nu}} \mathcal{L}_M \right) \delta g^{\mu\nu}\,\epsilon \\ &= \int \frac{1}{\sqrt{g}} \frac{\partial \left(\sqrt{g}\, \mathcal{L}_M \right)}{\partial g^{\mu\nu}} \delta g^{\mu\nu}\,\epsilon \end{aligned} \tag{1.18}$$

令 $-\frac{2}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\,\mathcal{L}_M\right)}{\partial g^{\mu\nu}} \left(dx^\mu\right)_a \left(dx^\nu\right)_b = T_{ab}$，最終得到，

$$\delta S_M = -\frac{1}{2} \int T_{ab} \delta g^{ab}\,\epsilon \tag{1.19}$$

1.3 作用量前兩項的變分

通過 1.1 和 1.2 節的計算，我們得到了作用量前兩項的變分，

$$\begin{aligned} \delta\left(S_{EH} + S_M\right) =& \frac{1}{2\kappa} \int \left(R_{ab} - \frac{1}{2} g_{ab} R + \Lambda \right) \delta g^{ab}\,\epsilon - \frac{1}{2} \int T_{ab} \delta g^{ab}\,\epsilon \\ &+ \frac{1}{2\kappa} \int \nabla_a \left(g^{bc} \delta \Gamma^a{}_{bc} - g^{ab} \delta \Gamma^c{}_{cb} \right) \epsilon \end{aligned} \tag{1.20}$$

其中，物質的能動量張量爲，

$$T_{ab} = -\frac{2}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\,\mathcal{L}_M\right)}{\partial g^{\mu\nu}} (dx^\mu)_a (dx^\nu)_b \tag{1.21}$$

在本筆記的最後，我們還會討論塵埃和電磁場的能動量張量。

2 邊界項

對 Einstein-Hilbert 作用量變分，我們會得到一個全散度的積分，

$$\begin{aligned} \mathcal{B} &= \frac{1}{2\kappa} \int \nabla_a \left(g^{bc} \delta \Gamma^a{}_{bc} - g^{ab} \delta \Gamma^c{}_{cb} \right) \epsilon \\ &= \frac{1}{2\kappa} \oint_{\partial M} \left(g^{bc} \delta \Gamma^a{}_{bc} - g^{ab} \delta \Gamma^c{}_{cb} \right) n_a\, \tilde{\epsilon} \end{aligned} \tag{2.1}$$

在邊界上，度規的變分爲零，而度規變分的微分只有沿超曲面切向的投影才爲零。因爲(2.1)含有仿射聯絡係數，所以并不等於零。

需要引入邊界項抵消 $\mathcal{B}$ 的作用，

$$S_B = -\int \nabla_a A^a \, \epsilon \tag{2.2}$$

其中，

$$A^a = g^{bc}\Gamma^a{}_{bc} - g^{ab}\Gamma^c{}_{cb} \tag{2.3}$$

2.1 邊界項中的矢量是否坐標依賴

坐標變換 $\{x^\mu\} \mapsto \{y^\mu\}$。下面先計算仿射聯絡係數的變換，

$$\begin{aligned}
\partial'_\nu g'_{\rho\sigma} &= \frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma}\partial_\kappa g_{\lambda\tau} + \frac{\partial x^\kappa}{\partial y^\nu}\partial_\kappa\left(\frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma}\right)g_{\lambda\tau} \\
&= \frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma}\partial_\kappa g_{\lambda\tau} + \frac{\partial}{\partial y^\nu}\left(\frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma}\right)g_{\lambda\tau} \\
&= \frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma}\partial_\kappa g_{\lambda\tau} + \left(\frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma} + \frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\sigma}\frac{\partial x^\tau}{\partial y^\rho}\right)g_{\lambda\tau}
\end{aligned} \tag{2.4}$$

帶入仿射聯絡係數的表達式，

$$\begin{aligned}
\Gamma'^\mu_{\nu\rho} &= g'^{\mu\sigma}\frac{1}{2}(\partial'_\nu g'_{\rho\sigma} + \partial'_\rho g'_{\nu\sigma} - \partial'_\sigma g'_{\nu\rho}) \\
&= \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\Gamma^\sigma_{\kappa\lambda} + g'^{\mu\sigma}g_{\lambda\tau}\frac{1}{2}\left(\frac{\partial}{\partial y^\nu}\left(\frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma}\right) + \frac{\partial}{\partial y^\rho}\left(\frac{\partial x^\lambda}{\partial y^\nu}\frac{\partial x^\tau}{\partial y^\sigma}\right) - \frac{\partial}{\partial y^\sigma}\left(\frac{\partial x^\lambda}{\partial y^\nu}\frac{\partial x^\tau}{\partial y^\rho}\right)\right) \\
&= \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\Gamma^\sigma_{\kappa\lambda} + g'^{\mu\sigma}g_{\lambda\tau}\frac{1}{2}\Bigg(\frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma} + \frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\sigma}\frac{\partial x^\tau}{\partial y^\rho} + \frac{\partial^2 x^\lambda}{\partial y^\rho \partial y^\nu}\frac{\partial x^\tau}{\partial y^\sigma} + \frac{\partial^2 x^\lambda}{\partial y^\rho \partial y^\sigma}\frac{\partial x^\tau}{\partial y^\nu} - \\
&\qquad \frac{\partial^2 x^\lambda}{\partial y^\sigma \partial y^\nu}\frac{\partial x^\tau}{\partial y^\rho} - \frac{\partial^2 x^\lambda}{\partial y^\sigma \partial y^\rho}\frac{\partial x^\tau}{\partial y^\nu}\Bigg) \\
&= \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\Gamma^\sigma_{\kappa\lambda} + g'^{\mu\sigma}g_{\lambda\tau}\frac{1}{2}\left(\frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma} + \frac{\partial^2 x^\lambda}{\partial y^\rho \partial y^\nu}\frac{\partial x^\tau}{\partial y^\sigma}\right) \\
&= \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\Gamma^\sigma_{\kappa\lambda} + g^{\kappa_1\kappa_2}\frac{\partial y^\mu}{\partial x^{\kappa_1}}\frac{\partial y^\sigma}{\partial x^{\kappa_2}}g_{\lambda\tau}\frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\rho}\frac{\partial x^\tau}{\partial y^\sigma} \\
&= \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\Gamma^\sigma_{\kappa\lambda} + g^{\kappa\tau}\frac{\partial y^\mu}{\partial x^\kappa}g_{\lambda\tau}\frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\rho} \\
&= \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial y^\nu}\frac{\partial x^\lambda}{\partial y^\rho}\Gamma^\sigma_{\kappa\lambda} + \frac{\partial y^\mu}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\rho}
\end{aligned} \tag{2.5}$$

帶入 A^a 分量的表達式，得到 A^μ 的變換關係，

$$\begin{aligned}
A'^\mu &= g'^{\nu\rho}\Gamma'^\mu_{\nu\rho} - g'^{\mu\nu}\Gamma'^\rho_{\rho\nu} \\
&= \frac{\partial y^\mu}{\partial x^\sigma}A^\sigma + g'^{\nu\rho}\frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial y^\nu \partial y^\rho} - g'^{\mu\nu}\frac{\partial y^\rho}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial y^\rho \partial y^\nu} \\
&= \frac{\partial y^\mu}{\partial x^\sigma}A^\sigma + g^{\kappa\lambda}\left(\frac{\partial y^\nu}{\partial x^\kappa}\frac{\partial y^\rho}{\partial x^\lambda}\frac{\partial y^\mu}{\partial x^\sigma} - \frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial y^\nu}{\partial x^\lambda}\frac{\partial y^\rho}{\partial x^\sigma}\right)\frac{\partial^2 x^\sigma}{\partial y^\nu \partial y^\rho}
\end{aligned} \tag{2.6}$$

注意到(2.6)的第 2 項，

$$\begin{aligned}
g^{\kappa\lambda}\left(\frac{\partial y^\nu}{\partial x^\kappa}\frac{\partial y^\rho}{\partial x^\lambda}\frac{\partial y^\mu}{\partial x^\sigma}\right)\frac{\partial^2 x^\sigma}{\partial y^\nu \partial y^\rho} &= g^{\kappa\lambda}\frac{\partial y^\nu}{\partial x^\kappa}\frac{\partial y^\rho}{\partial x^\lambda}\left(\frac{\partial}{\partial y^\nu}\left(\frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\rho}\right) - \frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial}{\partial y^\nu}\left(\frac{\partial y^\mu}{\partial x^\sigma}\right)\right) \\
&= -g^{\kappa\lambda}\frac{\partial y^\nu}{\partial x^\kappa}\frac{\partial y^\rho}{\partial x^\lambda}\frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial}{\partial y^\nu}\left(\frac{\partial y^\mu}{\partial x^\sigma}\right) \\
&= -g^{\kappa\lambda}\frac{\partial y^\rho}{\partial x^\lambda}\frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial^2 y^\mu}{\partial x^\kappa \partial x^\sigma} \\
&= -g^{\kappa\sigma}\frac{\partial^2 y^\mu}{\partial x^\kappa \partial x^\sigma}
\end{aligned} \tag{2.7}$$

以及(2.6)的第 3 項，

$$
\begin{aligned}
-g^{\kappa\lambda}\left(\frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial y^\nu}{\partial x^\lambda}\frac{\partial y^\rho}{\partial x^\sigma}\right)\frac{\partial^2 x^\sigma}{\partial y^\nu \partial y^\rho} &= -g^{\kappa\lambda}\frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial y^\nu}{\partial x^\lambda}\left(\frac{\partial}{\partial y^\nu}\left(\frac{\partial y^\rho}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\rho}\right)-\frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial}{\partial y^\nu}\left(\frac{\partial y^\rho}{\partial x^\sigma}\right)\right) \\
&= g^{\kappa\lambda}\frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial y^\nu}{\partial x^\lambda}\frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial}{\partial y^\nu}\left(\frac{\partial y^\rho}{\partial x^\sigma}\right) \\
&= g^{\kappa\lambda}\frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial^2 y^\rho}{\partial x^\lambda \partial x^\sigma} \\
&= g^{\kappa\lambda}\left(\frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial y^\rho}{\partial x^\lambda}\right)-\frac{\partial x^\sigma}{\partial y^\rho}\frac{\partial y^\rho}{\partial x^\lambda}\frac{\partial^2 y^\mu}{\partial x^\kappa \partial x^\sigma}\right) \\
&= g^{\kappa\lambda}\frac{\partial}{\partial y^\rho}\left(\frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial y^\rho}{\partial x^\lambda}\right)-g^{\kappa\sigma}\frac{\partial^2 y^\mu}{\partial x^\kappa \partial x^\sigma}
\end{aligned}
\tag{2.8}
$$

帶入有，

$$
A'^\mu = \frac{\partial y^\mu}{\partial x^\sigma}A^\sigma + g^{\kappa\sigma}\left(-2\frac{\partial^2 y^\mu}{\partial x^\kappa \partial x^\sigma}+\frac{\partial}{\partial y^\rho}\left(\frac{\partial y^\mu}{\partial x^\kappa}\frac{\partial y^\rho}{\partial x^\sigma}\right)\right) \tag{2.9}
$$

可見矢量場 A^a 是坐標依賴的。

2.2 邊界項的變分

我們希望通過對邊界項變分，抵消掉 $\mathcal{B}$，這是顯然可以達到的，實際上，

$$
\begin{aligned}
S_B &= -\int \nabla_a A^a \,\epsilon = -\oint A^a n_a \,\tilde{\epsilon} \\
\Longrightarrow \delta S_B &= -\oint_{\partial M} \delta A^a n_a \,\tilde{\epsilon} \\
&= -\mathcal{B} - \oint_{\partial M}\left(\Gamma^a{}_{bc}\delta g^{bc} - \Gamma^c{}_{cb}\delta g^{ab}\right) n_a \,\tilde{\epsilon} \\
&= -\mathcal{B}
\end{aligned}
\tag{2.10}
$$

2.3 Gibbons-Hawking-York 邊界項

2.3.1 一個比較麻煩的推導

邊界項在變分時只要僅相差 δg^{ab} 的倍數，實際上就是等同的。

Gibbons-Hawking-York 邊界項爲，

$$
S_{GHY} = \frac{1}{\kappa}\oint_{\partial M} K \,\tilde{\epsilon} \tag{2.11}
$$

對其進行變分，

$$
\delta S_{GHY} = \frac{1}{\kappa}\oint_{\partial M}\left(\delta K \,\tilde{\epsilon} + K \,\delta\tilde{\epsilon}\right) \tag{2.12}
$$

我們知道 $\sqrt{h} = \sqrt{g}/f$，那麼 $\delta\tilde{\epsilon}$ 將只與 δg^{ab} 有關，在邊界上這一項不帶來影響，所以，

$$
\delta S_{GHY} = \frac{1}{\kappa}\oint_{\partial M} \delta K \,\tilde{\epsilon} \tag{2.13}
$$

對於外曲率標量的變分，

$$
\delta K = \left(\nabla_a n_b\right)\delta g^{ab} - g^{ab} n_c \delta\Gamma^c{}_{ab} \tag{2.14}
$$

第一項在邊界爲零，所以，

$$
\delta S_{GHY} = -\frac{1}{\kappa}\oint_{\partial M} n_c g^{ab}\delta\Gamma^c{}_{ab}\,\tilde{\epsilon} \tag{2.15}
$$

對被積分的部分進一步化簡，

$$g^{ab}\delta\Gamma^c{}_{ab} = g^{ab}g^{cd}\frac{1}{2}\big(\partial_a\delta g_{bd} + \partial_b\delta g_{ad} - \partial_d\delta g_{ab}\big) + g^{ab}\frac{1}{2}\big(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\big)\delta g^{cd} \tag{2.16}$$

(2.16)的第二項在邊界處爲零，所以，

$$\begin{aligned}\delta S_{GHY} &= -\frac{1}{\kappa}\oint_{\partial M} g^{ab}g^{cd}\big(\partial_a\delta g_{bd} - \frac{1}{2}\partial_d\delta g_{ab}\big)n_c\,\tilde{\epsilon} \\ &= -\frac{1}{\kappa}\oint_{\partial M} n^c g^{ab}\big(\partial_a\delta g_{bc} - \frac{1}{2}\partial_c\delta g_{ab}\big)\,\tilde{\epsilon}\end{aligned} \tag{2.17}$$

對於 $\mathcal{B}$，即(2.1)，我們將被積分的部分進行化簡，

$$\begin{aligned}&g^{bc}\delta\Gamma^a{}_{bc} - g^{ab}\delta\Gamma^c{}_{cb} \\ =&g^{bc}g^{ad}\frac{1}{2}\big(\partial_b\delta g_{cd} + \partial_c\delta g_{bd} - \partial_d\delta g_{bc}\big) + g^{bc}\frac{1}{2}\big(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\big)\delta g^{ad} \\ &- g^{ab}g^{cd}\frac{1}{2}\big(\partial_b\delta g_{cd} + \partial_c\delta g_{bd} - \partial_d\delta g_{bc}\big) - g^{ab}\frac{1}{2}\big(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\big)\delta g^{ad}\end{aligned} \tag{2.18}$$

(2.18)的第二、四項僅含 δg^{ab}，在邊界處爲零，所以，

$$\begin{aligned}\mathcal{B} &= \frac{1}{2\kappa}\oint_{\partial M}\Big(g^{bc}g^{ad}\frac{1}{2}\big(\partial_b\delta g_{cd} + \partial_c\delta g_{bd} - \partial_d\delta g_{bc}\big) - g^{ab}g^{cd}\frac{1}{2}\big(\partial_b\delta g_{cd} + \partial_c\delta g_{bd} - \partial_d\delta g_{bc}\big)\Big)n_a\,\tilde{\epsilon} \\ &= \frac{1}{2\kappa}\oint_{\partial M} g^{ab}g^{cd}\big(-\partial_b\delta g_{cd} + \partial_d\delta g_{bc}\big)n_a\,\tilde{\epsilon} \\ &= \frac{1}{2\kappa}\oint_{\partial M} n^a g^{bc}\big(-\partial_a\delta g_{bc} + \partial_c\delta g_{ab}\big)\,\tilde{\epsilon}\end{aligned} \tag{2.19}$$

我們會得到，

$$\delta S_{GHY} + \mathcal{B} = -\frac{1}{2\kappa}\oint_{\partial M} n^c g^{ab}\big(\partial_a\delta g_{bc}\big)\,\tilde{\epsilon} \tag{2.20}$$

由於在邊界上 δg_{ab} 恆爲零，所以，

$$h^d{}_a\partial_d\delta g_{bc} = 0 \tag{2.21}$$

$$\begin{aligned}&\Longrightarrow \partial_a\delta g_{bc} = \epsilon\, n^d n_a\partial_d\delta g_{bc} \\ &\Longrightarrow n^c g^{ab}\partial_a\delta g_{bc} = \epsilon n^c n^d n^b\partial_d\delta g_{bc}\end{aligned} \tag{2.22}$$

那麼，

$$\delta S_{GHY} + \mathcal{B} = -\frac{1}{2\kappa}\oint_{\partial M}\epsilon n^c n^d n^b\partial_d\delta g_{bc}\,\tilde{\epsilon} \tag{2.23}$$

下面我們來證明這個積分是零：利用 $n^a n^b\nabla_a n_b = 0$，

$$\begin{aligned}&0 = \oint_{\partial M}\delta(n^a n^b\nabla_a n_b)\,\tilde{\epsilon} \\ \Longrightarrow\;&0 = \oint_{\partial M} n^a n^b\big(-n_c\delta\Gamma^c{}_{ab}\big)\,\tilde{\epsilon} \\ \Longrightarrow\;&0 = \oint_{\partial M} -n^a n^b n_c\frac{1}{2}g^{cd}\big(\partial_a\delta g_{bd} + \partial_b\delta g_{ad} - \partial_d\delta g_{ab}\big)\,\tilde{\epsilon} \\ \Longrightarrow\;&0 = \oint_{\partial M} -\frac{1}{2}n^a n^b n^d\big(\partial_a\delta g_{bd} + \partial_b\delta g_{ad} - \partial_d\delta g_{ab}\big)\,\tilde{\epsilon} \\ \Longrightarrow\;&0 = \oint_{\partial M} -\frac{1}{2}n^a n^b n^d\big(\partial_a\delta g_{bd}\big)\,\tilde{\epsilon}\end{aligned} \tag{2.24}$$

得證。

所以，

$$\delta S_{GHY} + \mathcal{B} = 0 \tag{2.25}$$

2.3.2 更簡單直接的推導

還是考慮 Gibbons-Hawking-York 邊界項的變分，

$$\begin{aligned}\delta S_{GHY} &= \frac{1}{\kappa}\oint_{\partial M} \delta K\,\tilde{\epsilon} \\ &= \frac{1}{\kappa}\oint_{\partial M} \delta(\nabla^a n_a)\,\tilde{\epsilon} \\ &= \frac{1}{\kappa}\oint_{\partial M} \delta\big((g^{ab}-\epsilon n^a n^b)\nabla_a n_b\big)\,\tilde{\epsilon} \\ &= \frac{1}{\kappa}\oint_{\partial M} (g^{ab}-\epsilon n^a n^b)\big(-n_c\delta\Gamma^c{}_{ab}\big)\,\tilde{\epsilon}\end{aligned} \tag{2.26}$$

其中用到了 $n^b\nabla_a n_b = 0$，(爲了避免與 h_{ab} 在超曲面上的逆變形式 h^{ab} 混淆，我們不使用通常文獻中的符號系統，即 $h^{ab} = h^a{}_c g^{bc}$)。進一步化簡被積分的部分，

$$\begin{aligned}&(g^{ab}-\epsilon n^a n^b)\big(-n_c\delta\Gamma^c{}_{ab}\big) \\ =& -n_c(g^{ab}-\epsilon n^a n^b)\frac{1}{2}g^{cd}\big(\partial_a\delta g_{bd}+\partial_b\delta g_{ad}-\partial_d\delta g_{ab}\big)\end{aligned} \tag{2.27}$$

注意 δg_{ab} 的導數向超曲面做投影后爲零，即(2.21)，最終有，

$$\begin{aligned}&(g^{ab}-\epsilon n^a n^b)\big(-n_c\delta\Gamma^c{}_{ab}\big) \\ =& n_c(g^{ab}-\epsilon n^a n^b)\frac{1}{2}g^{cd}\big(\partial_d\delta g_{ab}\big)\end{aligned} \tag{2.28}$$

所以，

$$\delta S_{GHY} = \frac{1}{\kappa}\oint_{\partial M} \frac{1}{2}n^c(g^{ab}-\epsilon n^a n^b)\big(\partial_c\delta g_{ab}\big) \tag{2.29}$$

與(2.19)對比，并注意到(2.22)，直接得到，

$$\delta S_{GHY} + \mathcal{B} = 0 \tag{2.30}$$

3 物質場的能動量張量

根據 1.2 節的推導，物質場的能動量張量爲，

$$T_{ab} = \left(-2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M\right)(dx^\mu)_a(dx^\nu)_b = -\frac{2}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\,\mathcal{L}_M\right)}{\partial g^{\mu\nu}}(dx^\mu)_a(dx^\nu)_b \tag{3.1}$$

下面的推導會用到一些我自創的符號。

3.1 塵埃的能動量張量

塵埃的作用量爲，

$$S_{dust} = \int \frac{\rho_M}{\gamma}c^2\,\epsilon \tag{3.2}$$

其中 ρ_M 是塵埃的密度，

$$\rho_M = \frac{dm}{\tilde{\epsilon}} \tag{3.3}$$

$\tilde{\epsilon}$ 是時空 3+1 分解后的空間體元，dm 是體元内物質的靜質量。而 γ 是物質速度的函數，

$$\gamma = \frac{1}{\sqrt{1-\frac{v^i v_i}{c^2}}} \tag{3.4}$$

更嚴謹的寫法是，在與空間超曲面“適配”的坐標 $\{t, x^i\}$ 下，

$$\gamma = \frac{1}{c} U^a (dt)_a = \frac{dt}{d\tau} \tag{3.5}$$

其中 U^a 是塵埃的四維速度，$U^a = \left(\frac{\partial}{\partial \tau}\right)^a$；$d\tau$ 是塵埃的固有時間的微分。

下面來分別計算 ρ_M 和 γ 的變分：

首先，對於 ρ_M，

$$\delta\rho_M = dm\delta\frac{1}{\tilde{\epsilon}} = -\rho_M\frac{\delta\tilde{\epsilon}}{\tilde{\epsilon}} \tag{3.6}$$

而對於 $\delta\tilde{\epsilon}$，代入(1.8)，

$$\delta\tilde{\epsilon} = -\frac{1}{2} g_{\mu\nu}\delta g^{\mu\nu}\tilde{\epsilon} \tag{3.7}$$

所以，

$$\delta\rho_M = \frac{1}{2}\rho_M g_{\mu\nu}\delta g^{\mu\nu} \tag{3.8}$$

其次，對於 γ，我們在坐標 $\{t, x^i\}$ 下計算，

$$\delta\gamma = -\gamma\frac{\delta d\tau}{d\tau} \tag{3.9}$$

其中，

$$\begin{aligned}
\delta d\tau &= \frac{1}{c}\sqrt{(g_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu} - d\tau \\
&= \frac{1}{c^2}\frac{dx^\mu dx^\nu}{2d\tau}\delta g_{\mu\nu} \\
&= \frac{1}{2c^2} d\tau U^\mu U^\nu \delta g_{\mu\nu}
\end{aligned} \tag{3.10}$$

代入，并利用(1.6)，得到，

$$\delta\gamma = -\gamma\frac{1}{2c^2}U^\mu U^\nu\delta g_{\mu\nu} = \gamma\frac{1}{2c^2}U_\mu U_\nu\delta g^{\mu\nu} \tag{3.11}$$

綜上，我們有，

$$\begin{aligned}
\delta\mathcal{L}_M &= \left(\frac{\delta\rho_M}{\gamma} + \rho_M\delta\frac{1}{\gamma}\right)c^2 \\
&= \left(\frac{\delta\rho_M}{\gamma} - \frac{\rho_M}{\gamma^2}\delta\gamma\right)c^2 \\
&= \left(\frac{1}{2}\frac{\rho_M}{\gamma}g_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2c^2}\frac{\rho_M}{\gamma}U_\mu U_\nu\delta g^{\mu\nu}\right)c^2
\end{aligned} \tag{3.12}$$

所以，$\mathcal{L}_M$ 對 $g^{\mu\nu}$ 的偏導數爲，

$$\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} = \frac{1}{2}\frac{\rho_M}{\gamma}\left(c^2 g_{\mu\nu} - U_\mu U_\nu\right) \tag{3.13}$$

代入(3.1)，得到塵埃的能動量張量，

$$T_{ab} = \frac{\rho_M}{\gamma}U_a U_b \tag{3.14}$$

3.2 電磁場的能動量張量

電磁場（和其中的帶電物質）的作用量爲，

$$S_{EM} = \int\left(\frac{\rho_M}{\gamma}c^2 + \frac{\rho_Q}{\gamma}A^a U_a - \frac{1}{4\mu_0}F^{ab}F_{ab}\right)\epsilon \tag{3.15}$$

其中 ρ_Q 是帶電物質的電荷密度，

$$\rho_Q = \frac{dq}{\tilde{\epsilon}} \tag{3.16}$$

A^a 是四維矢勢，在與空間超曲面“適配”的坐標 $\{t, x^i\}$ 下，

$$\begin{cases} A_a\Big(\dfrac{\partial}{\partial t}\Big)^a = A_0 = -\dfrac{\phi}{c} \\ A_a\Big(\dfrac{\partial}{\partial x^i}\Big)^a = A_i \end{cases} \tag{3.17}$$

這裏要注意，四維矢勢的協變分量 A_a 是與度規無關的，而其逆變分量則應視爲 $A^a = g^{ab}A_b$，是與度規有關的。具體理由在下面計算第二項的變分時給出。另外，在實際計算中，我們也確實習慣將 $(dx^\mu)_a$ 視爲坐標基矢，將四維矢勢寫爲 $A_\mu(dx^\mu)_a$。最後，$F_{ab} = (dA)_{ab} = 2\nabla_{[a}A_{b]}$ 是四維電磁場張量，在坐標 $\{t, x^i\}$ 下，

$$F_{ab}\Big(\frac{\partial}{\partial x^\mu}\Big)^a\Big(\frac{\partial}{\partial x^\nu}\Big)^b = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_2/c & B_2 & -B_1 & 0 \end{pmatrix} \tag{3.18}$$

下面來計算 $\delta\mathcal{L}_M$：

首先計算第二項，在 3.1 節裏已經計算了 $\frac{\rho_M}{\gamma}$ 的變分，類似地，

$$\delta\frac{\rho_Q}{\gamma} = \frac{1}{2}\frac{\rho_Q}{\gamma}\Big(g_{\mu\nu} - \frac{1}{c^2}U_\mu U_\nu\Big)\delta g^{\mu\nu} \tag{3.19}$$

而對於 A^aU_a，四維矢勢不隨度規變化，但是四維速度含有 $d\tau$，代入(3.10)，

$$\begin{aligned} \delta U^\mu &= -U^\mu\frac{\delta d\tau}{d\tau} \\ &= U^\mu\frac{1}{2c^2}U_\nu U_\rho\delta g^{\nu\rho} \end{aligned} \tag{3.20}$$

那麼，如果認爲 A^a 與度規無關（這是錯誤的），

$$\begin{aligned} \delta(A^aU_a) &= g_{ab}A^a\delta U^b + A^aU^b\delta g_{ab} \\ &= A^aU_a\frac{1}{2c^2}U_\mu U_\nu\delta g^{\mu\nu} - A_aU_b\delta g^{ab} \end{aligned} \tag{3.21}$$

所以，

$$\begin{aligned} \delta\Big(\frac{\rho_Q}{\gamma}A^aU_a\Big) &= \frac{1}{2}\frac{\rho_Q}{\gamma}A^aU_a\Big(g_{\mu\nu} - \frac{1}{c^2}U_\mu U_\nu\Big)\delta g^{\mu\nu} + \frac{\rho_Q}{\gamma}\Big(A_cU^c\frac{1}{2c^2}U_aU_b - A_aU_b\Big)\delta g^{ab} \\ &= \frac{\rho_Q}{\gamma}\Big(A^cU_c\frac{1}{2}g_{ab} - A_{(a}U_{b)}\Big)\delta g^{ab} \end{aligned} \tag{3.22}$$

實際上，第二項的變分應該等於 $\frac{\rho_Q}{\gamma}A^cU_c\frac{1}{2}g_{ab}\delta g^{ab}$，這裏多出來了一項 $-\frac{\rho_Q}{\gamma}A_{(a}U_{b)}\delta g^{ab}$。這說明，我們確實應該認識到 A_a 才是與度規無關的，而 A^a 則是 $g^{ab}A_b$，是與度規有關的。

其次，我們計算第三項的變分，這裏顯然也是協變分量與度規無關，所以，

$$\delta\Big(\frac{1}{4\mu_0}F^{ab}F_{ab}\Big) = \frac{1}{2\mu_0}F_{ab}F^a{}_c\delta g^{bc} \tag{3.23}$$

綜上所述，電磁場（和其中的帶電物質）的能動量張量爲，

$$\begin{aligned} T_{ab} =& -2\Big(\frac{1}{2}\frac{\rho_M}{\gamma}\left(c^2g_{ab} - U_aU_b\right) + \frac{\rho_Q}{\gamma}A^cU_c\frac{1}{2}g_{ab} - \frac{1}{2\mu_0}F_{ca}F^c{}_b\Big) \\ &+ g_{ab}\Big(\frac{\rho_M}{\gamma}c^2 + \frac{\rho_Q}{\gamma}A^cU_c - \frac{1}{4\mu_0}F^{cd}F_{cd}\Big) \\ =&\frac{\rho_M}{\gamma}U_aU_b + \frac{1}{\mu_0}F_{ca}F^c{}_b - \frac{1}{4\mu_0}g_{ab}F^{cd}F_{cd} \end{aligned} \tag{3.24}$$

在平直時空，坐標 $\{t, x^i\}$ 下的能動量張量分量爲，

$$T^{\mu\nu} = \frac{\rho_M}{\gamma} U^\mu U^\nu + \begin{pmatrix} W & \vec{S}/c \\ \vec{S}/c & W\delta_{ij} - \frac{1}{\mu_0} B_i B_j - \epsilon_0 E_i E_j \end{pmatrix} \tag{3.25}$$

其中，

$$\begin{cases} W = \dfrac{1}{2}\Big(\dfrac{1}{\mu_0} B^2 + \epsilon_0 E^2\Big) \\ \vec{S} = \dfrac{1}{\mu_0} \vec{E} \times \vec{B} \end{cases} \tag{3.26}$$